SOLUTIONS TO EXAM 3, MATH 10550

1. *In finding an approximate solution to the equation $x^4 + x - 2 = 0$ using Newton's Method with initial approximation $x_1 = -1$, what is x_2?*

Solution: Let $f(x) = x^4 + x - 2$, then $f(-1) = -2$ and $f'(x) = 4x^3 + 1$ so $f'(-1) = -3$. Recall that Newton's method says

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Using $n = 1$, we have $x_2 = -1 - \frac{-2}{-3} = \frac{-5}{3}$

2. *If $f'(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$ and $f(1) = \frac{8}{3}$, find $f(4)$.*

Solution: $f(x) = \int \sqrt{x} + \frac{1}{\sqrt{x}} dx = \int x^{1/2} + x^{-1/2} dx = \frac{2}{3}x^{3/2} + 2x^{1/2} + C$. Now use $f(1) = \frac{8}{3}$ to solve for C, you should get $C = 0$. Therefore, $f(x) = \frac{2}{3}x^{3/2} + 2x^{1/2}$ and $f(4) = \frac{2}{3}4^{3/2} + 2*4^{1/2} = \frac{28}{3}$.

3. *Calculate the following indefinite integral*

$$\int (4 - 3t^2)(4t + 1) dt.$$

Solution: Start by distributing to get

$$\int (4 - 3t^2)(4t + 1) dt = \int 16t + 4 - 12t^3 - 3t^2 dt$$

$$= 8t^2 + 4t - 3t^4 - t^3 + C = -3t^4 - t^3 + 8t^2 + 4t + C.$$

4. *Let $f(x) = \int_{\sin x}^{2} \frac{1}{\sqrt{t^4 + 6}} dt$. What is $f'(x)$?*

Solution: Let $g(x) = \int_{2}^{x} \frac{1}{\sqrt{t^4 + 6}} dt$, then Fundamental Theorem of Calculus (FTC) says that

$$g'(x) = \frac{1}{\sqrt{x^4 + 6}}.$$

Then $g(\sin(x)) = \int_{2}^{\sin(x)} \frac{1}{\sqrt{t^4 + 6}}$ and $f(x) = -g(\sin(x))$. Therefore, by the chain rule, we have $f'(x) = -\cos(x)g'(\sin(x))$. Using what we found above about $g'(x)$, we get

$$f'(x) = \frac{-\cos(x)}{\sqrt{\sin^4(x) + 6}}$$

5. *Evaluate the indefinite integral*

$$\int \frac{\sec^2(\sqrt{x})}{\sqrt{x}} dx.$$

Solution: Use u-substitution and let $u = \sqrt{x}$, then $du = \frac{1}{2\sqrt{x}}dx$ and hence $2du = \frac{1}{\sqrt{x}}dx$. Therefore,

$$\int \frac{\sec^2(\sqrt{x})}{\sqrt{x}}dx = \int 2\sec^2(u)du = 2\tan(u) + C = 2\tan(\sqrt{x}) + C.$$

6. *Evaluate the definite integral* $\int_0^1 x^2(\sqrt{x} + 3)dx$.

Solution:

$$\int_0^1 x^2(\sqrt{x} + 3)dx = \int_0^1 x^{5/2} + 3x^2 dx$$

$$= \frac{2}{7}x^{7/2} + x^3 \Big|_0^1 = (\frac{2}{7} + 1) - (0 + 0) = \frac{9}{7}.$$

7. *Calculate the definite integral*

$$\int_{-1}^2 |x|dx.$$

Solution: Recall that $|x| = -x$ if $x < 0$ and $|x| = x$ if $x \geq 0$, therefore we can rewrite this integral as

$$\int_{-1}^2 |x|dx = \int_{-1}^0 -x dx + \int_0^2 x dx = \frac{-x^2}{2} \Big|_{-1}^0 + \frac{x^2}{2} \Big|_0^2 = (0 - \frac{-1}{2}) + (2 - 0) = \frac{5}{2}.$$

8. *Evaluate the definite integral*

$$\int_0^\pi (2x + 1)\sin(x^2 + x)dx.$$

Solution: Let $u = x^2 + x$, then $du = 2x + 1$. So

$$\int_0^\pi (2x + 1)\sin(x^2 + x)dx = \int_0^{\pi^2 + \pi} \sin(u)du = -\cos(u)\Big|_0^{\pi^2 + \pi}$$

$$= -\cos(\pi^2 + \pi) - (-\cos(0)) = 1 - \cos(\pi^2 + \pi).$$

9. *Find the equation of the slant asymptote of the curve*

$$y = \frac{2x^3 - 3x^2 + 4x + 6}{x^2 + 2x}.$$

Solution: Use long division to find that

$$\frac{2x^3 - 3x^2 + 4x + 6}{x^2 + 2x} = 2x - 7 + \frac{18x + 6}{x^2 + 2x}.$$

Therefore, the slant asymptote is $y = 2x - 7$.

10. *Let* $A = \int_0^1 \frac{x}{x + 1}dx$. *Which of the following is true of A?*

Solution: We know that if M is the maximum value of $f(x)$ on the interval $[a, b]$ and m is the minimum value of $f(x)$ on that interval, then

$$m(b - a) \le \int_a^b f(x)dx \le M(b - a).$$

Let $f(x) = \frac{x}{x+1}$ and $a = 0$, $b = 1$. Then we need to find the maximum values of f on the interval $[0, 1]$. We see that $f'(x) = \frac{1}{(x+1)^2}$, therefore $f(x)$ has no critical values on the interval $[0, 1]$ meaning the max/mins can only occur at the endpoints. $f(0) = 0$ and $f(1) = \frac{1}{2}$, so $m = 0$ and $M = \frac{1}{2}$. Plugging into the formula form above gives:

$$0(1 - 0) \le \int_0^1 \frac{x}{x + 1}dx \le \frac{1}{2}(1 - 0)$$

In particular, $0 \le A \le \frac{1}{2}$.

11. *Using the definition of the definite integral, evaluate*

$$\int_0^2 1 + x^2 dx.$$

Hint:

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n + 1)(2n + 1).$$

Solution: Notice that $\Delta x = \frac{2}{n}$ and $x_i = \frac{2i}{n}$. Therefore,

$$\int_0^2 1 + x^2 dx = \lim_{n \to \infty} \frac{2}{n} \sum_{i=1}^n f(x_i) = \lim_{n \to \infty} \frac{2}{n} \sum_{i=1}^n \left[1 + (\frac{2i}{n})^2 \right] = \lim_{n \to \infty} \frac{2}{n} \left(\sum_{i=1}^n 1 + \sum_{i=1}^n \frac{4i^2}{n^2} \right).$$

$$= \lim_{n \to \infty} \frac{2}{n} \left(\sum_{i=1}^n 1 + \frac{4}{n^2} \sum_{i=1}^n i^2 \right).$$

Now, notice that $\sum_{i=1}^n 1 = n$ and $\sum_{i=1}^n i^2 = \frac{1}{6}n(n + 1)(2n + 1)$. Plugging this back into the above equation, we see that

$$\int_0^2 1 + x^2 dx = \lim_{n \to \infty} \frac{2}{n} \left(n + \frac{4}{n^2} \frac{1}{6}n(n + 1)(2n + 1) \right)$$

$$= \lim_{n \to \infty} 2 + \frac{8n(n + 1)(2n + 1)}{6n^3} = 2 + \frac{16}{6} = \frac{14}{3}$$

Verify your result using the Fundamental Theorem of Calculus.

Solution: $\int_0^2 1 + x^2 dx = x + \frac{x^3}{3} \Big|_0^2 = (2 + \frac{9}{3}) - 0 = \frac{14}{3}$

12. *A ball is thrown upward from a height of 32 feet above the ground on the planet Calculuto, with an initial velocity of 12 feet per second. The velocity of the ball at time t is $12 - 4t$ feet per second.*

(1) Find $s(t)$, the function giving the height of the ball at time t.

Solution: Note that $s(0) = 32$ and $v(t) = 12 - 4t$. Then we know that

$$s(t) = \int v(t)dt = \int 12 - 4t dt = 12t - 2t^2 + C.$$

Use $s(0) = 32$ to find that $C = 32$, hence $s(t) = -2t^2 + 12t + 32$.

(2) How long will the ball take to reach the ground?
Solution: The ball will reach the ground when $s(t) = 0$, so solve $0 = -2t^2 + 12t + 32 = -2(t - 8)(t + 2)$. Therefore $t = -2$ or 8, but time must be positive so it takes 8 seconds for the ball to reach the ground.

(3) How high will the ball go?
Solution: This is asking for the maximum height of the ball. To find this, we must find where the velocity is zero since this is the time when the maximum height will occur. $0 = v(t) = 12 - 4t$ implies that $t = 3$ is when the max occurs. Therefore the maximum height is $s(3) = -2(9) + 12(3) + 32 = 50$ feet.

13. *An open rectangular box (a box with no top) with square base and with a volume of 27 cubic feet is needed. Material for the base costs \$2 per square foot, and material for the sides costs \$1 per square foot. Determine the dimensions of the box that will minimize the cost of materials. Justify that your answer is a minimum.*
Solution: Let x be the length of each side of the base and y be the height of the box. Then the volume is given by $27 = x^2y$ solving for y gives $y = \frac{27}{x^2}$.

Now, the cost for the box is $C = 2x^2 + 4xy$. We rewrite this in terms of x using the fact that $y = \frac{27}{x^2}$:

$$C(x) = 2x^2 + \frac{108}{x}.$$

We want to minimize $C(x)$, so we should find the critical points. $C'(x) = 4x - \frac{108}{x^2}$. $C'(x) = 0$ if $x = 3$ and C' is undefined if $x = 0$ (but that won't make sense as a length for a side of a box).

To see that $x = 3$ is a minimum in the interval $(0, \infty)$, we know that $C(x)$ is continuous and differentiable in that interval and we have that $C'(x) < 0$ if $x < 3$ and $C'(x) > 0$ if $x > 3$. Therefore, there is a global minimum of the cost function on the interval $(0, \infty)$ when $x = 3$. If $x = 3$, then $y = \frac{27}{3^2} = 3$. So the box should be 3 feet x 3 feet x 3 feet.